A K-Matrix Tutorial

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Outline

- Why
- The Formalism.
- Simple Examples.
- Recent Analyses.

Note: See S. U. Chung, *et al.*, *Partial wave analysis in K-matrix formalism*, Ann. der Physik **4**:404,(1995).

What is the K-matrix

- In Partial wave analysis (PWA), resonances are often parameterized as Breit-Wigners.

$$BW(m) = \frac{m_0 \Gamma_0}{m_0^2 - m^2 - im\Gamma m}$$

$$\Gamma(m) \sim \left(\frac{m_0}{m}\right) \left(\frac{\rho(m)}{\rho(m_0)}\right) \left(\frac{F_l(q)}{F_l(q_0)}\right) \Gamma_0$$

- This approximation assumes an isolated resonance with a single measured decay.

What is the K-matrix

- If there is more than one resonance in the same partial wave that strongly overlap.
- The Scalar Meson Sector (all couple to $\pi\pi$ final states).

$$\begin{array}{rcc} f_0(600) & m = 400 - 1200\,MeV & \Gamma = 600 - 1000\,MeV \\ f_0(980) & m = 980\,MeV & \Gamma = 40 - 100\,MeV \\ f_0(1370) & m = 1200 - 1500\,MeV & \Gamma = 200 - 500 MeV \\ f_0(1500) & m = 1507\,MeV & \Gamma = 109\,MeV \\ f_0(1710) & m = 1718\,MeV & \Gamma = 137\,MeV \end{array}$$

- Broadly overlapping states.

- Decays overlap as well:

$$\begin{aligned} f_0(600) &\rightarrow \pi\pi \\ f_0(980) &\rightarrow \pi\pi, K\bar{K} \\ f_0(1370) &\rightarrow \pi\pi, K\bar{K}, \eta\eta, 4\pi \\ f_0(1500) &\rightarrow \pi\pi, K\bar{K}, \eta\eta, \eta\eta\prime, 4\pi \\ f_0(1710) &\rightarrow \pi\pi, K\bar{K}, \eta\eta \end{aligned}$$

- Lots of common decay modes.

Formalism

- Start with a scattering amplitude to connect an initial state to a final state.

$$S_{fi} \;=\; < f \mid S \mid i >$$

The scattering operator, S, is unitary: $SS^\dagger = I$.

- The transition operator, T, can be defined via

$$S \;=\; I + 2iT$$

- This yields an expression:

$$\left(T^\dagger\right)^{-1} - T^{-1} \;=\; 2iI$$

$$\left(T^{-1} + iI\right)^\dagger \;=\; \left(T^{-1} + iI\right)$$

- This yields a quantity which is Hermitian.

- We define the K operator in terms of the Hermitian combination:

$$K^{-1} \;\; = \;\; \left(T^{-1} + iI\right)$$

such that K is also Hermitian, $K = K^{\dagger}$.

- Time reversal of S and T leads to K also being symmetric. Thus, the K-operator, or the *K-matrix* can be chosen to be real and symmetric.

Formalism

- In terms of K, we have that:

$$T = K\left(I - iK\right)^{-1}$$

- We also have that S is

$$S = \left(I + iK\right)\left(I - iK\right)^{-1}$$

Formalism

- We define the K operator in terms of the Hermitian combination:

$$K^{-1} = \left(T^{-1} + iI\right)$$

  such that K is also Hermitian, $K = K^{\dagger}$.
- Time reversal of S and T leads to K also being symmetric. Thus, the K-operator, or the *K-matrix* can be chosen to be real and symmetric.

The K-matrix can be written as the sum of poles, m_α, and decay channels, i and j,

$$K_{ij} = \sum_\alpha \frac{g_{\alpha i} g_{\alpha_j}}{(m_\alpha^2 - m^2)\sqrt{\rho_i \rho_j}} .$$

The decay couplings are given as

$$\begin{aligned} g_{\alpha i}^2(m) &= m_\alpha \Gamma_{\alpha i}(m) \\ \Gamma_{\alpha i}(m) &= \gamma_{\alpha i}^2 \Gamma_\alpha^0 \rho_i (BF)^2 \end{aligned}$$

and ρ_i is the phase space for the specified decay. This yields an matrix whose dimensions is the number of decay modes.

Formalism

- If $S = e^{2i\delta}$, then $T = e^{i\delta} \sin\delta$ and the S-wave cross section is given as

$$\sigma = \left(\frac{4\pi}{q_i^2}\right) \sin^2\delta$$

The K-matrix can be shown to be:

$$K = \tan\delta$$

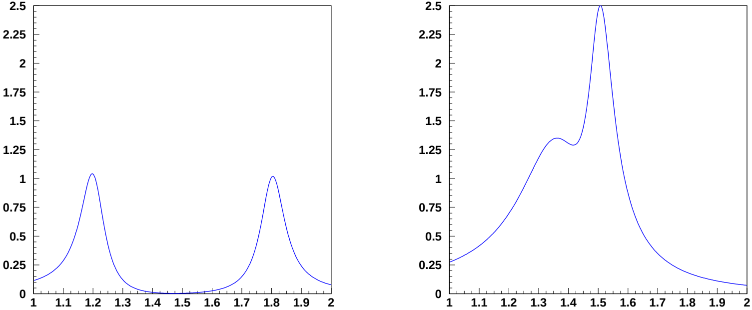
Two-pole K-Matrix

- Consider two resonances in the $J^{PC} = 0^{++}$ channel. We will consider two cases:

$$f_0(1200, \Gamma = 100) \qquad f_0(1800, \Gamma = 100)$$
$$f_0(1350, \Gamma = 300) \qquad f_0(1500, \Gamma = 100)$$

- In the first case, the resonances are relatively well isolated. In the second, there is very strong overlap.

Two-pole K-Matrix



- Breit-Wigner parameterization for the mass of the resonances.
(left) $f_0(1200, \Gamma = 100) \& f_0(1800, \Gamma = 100)$ and
(right) $f_0(1350, \Gamma = 300) \& f_0(1500, \Gamma = 100)$.

Two-pole K-Matrix

$$
\begin{aligned}
K_{ij}(m) &= \frac{m_1\Gamma_1(m)}{m_1^2 - m^2} + \frac{m_2\Gamma_2(m)}{m_2^2 - m^2} \\
\Gamma_i(m) &= \Gamma_i^0 \left(\frac{m_i}{m}\right)\left(\frac{q}{q_i}\right)
\end{aligned}
$$

$$
\begin{aligned}
T &= K\left(1 - iK\right)^{-1} \\
T &= \frac{m_1\Gamma_1(m)}{(m_1^2 - m^2) - im_1\Gamma_1(m) - i\frac{(m_1^2-m^2)}{(m_2^2-m^2)}m_2\Gamma_2(m)} \\
&+ \frac{m_2\Gamma_2(m)}{(m_2^2 - m^2) - im_2\Gamma_2(m) - i\frac{(m_2^2-m^2)}{(m_1^2-m^2)}m_1\Gamma_1(m)}
\end{aligned}
$$

Two-pole K-Matrix

- For the case of m_1 and m_2 very far apart, the terms like

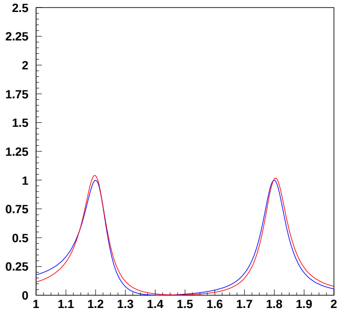
$$\sim \quad i\frac{(m_1^2 - m^2)}{(m_2^2 - m^2)} m_2 \Gamma_2(m)$$

are driven to zero far away from the main resonance. Thus, we get that

$$T \quad \approx \quad \frac{m_1 \Gamma_1(m)}{(m_1^2 - m^2) - i m_1 \Gamma_1(m)} + \frac{m_2 \Gamma_2(m)}{(m_2^2 - m^2) - i m_2 \Gamma_2(m)}$$

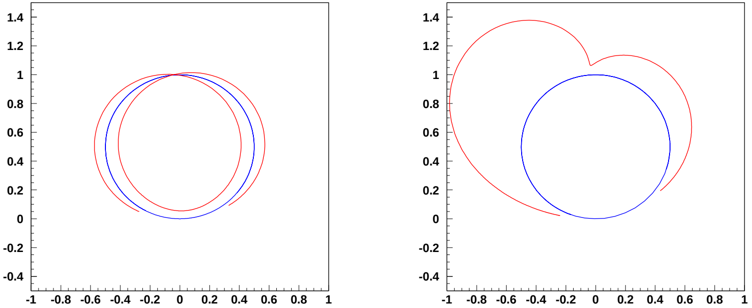
which looks like two Breit Wigner functions.

Two-pole K-matrix



- Breit-Wigner parameterization (red) and K-Matrix (blue) for the mass of the resonances, (left) $f_0(1200, \Gamma = 100)\&f_0(1800, \Gamma = 100)$ and (right) $f_0(1350, \Gamma = 300)\&f_0(1500, \Gamma = 100)$.

Two-pole K-Matrix



- Breit-Wigner parameterization (red) and K-Matrix (blue) for the Argand diagrams for the two resonances, (left) $f_0(1200, \Gamma = 100) \& f_0(1800, \Gamma = 100)$ and (right) $f_0(1350, \Gamma = 300) \& f_0(1500, \Gamma = 100)$.

One-pole, 2-decay K-Matrix

- Consider the $a_0(980)$, a $J^{PC} = 0^{++}$ resonance that couples to both $\eta\pi$ and $K\bar{K}$.
- The $K\bar{K}$ occurs near the peak of the resonance, (987.3 MeV).

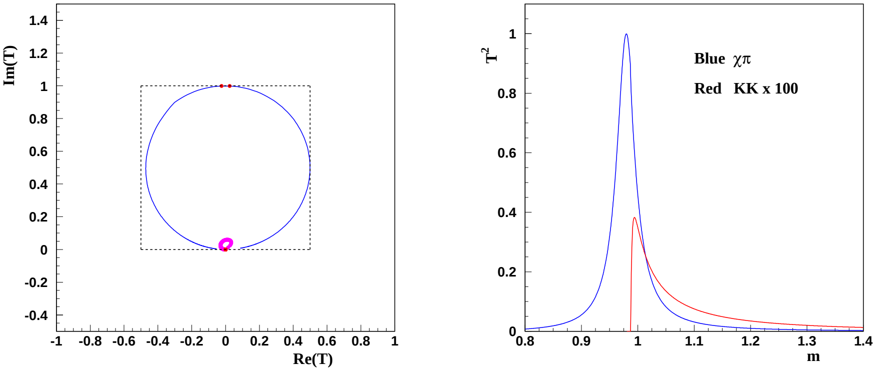
$$
\begin{aligned}
K_{11} &= \frac{\gamma_1^2 m_0 \Gamma_0}{m_0^2 - m^2} \\
K_{22} &= \frac{\gamma_2^2 m_0 \Gamma_0}{m_0^2 - m^2} \\
K_{12} &= \frac{\gamma_1 \gamma_2 m_0 \Gamma_0}{m_0^2 - m^2}
\end{aligned}
$$

with $\gamma_1^2 + \gamma_2^2 = 1$.

One-pole, 2-decay K-Matrix

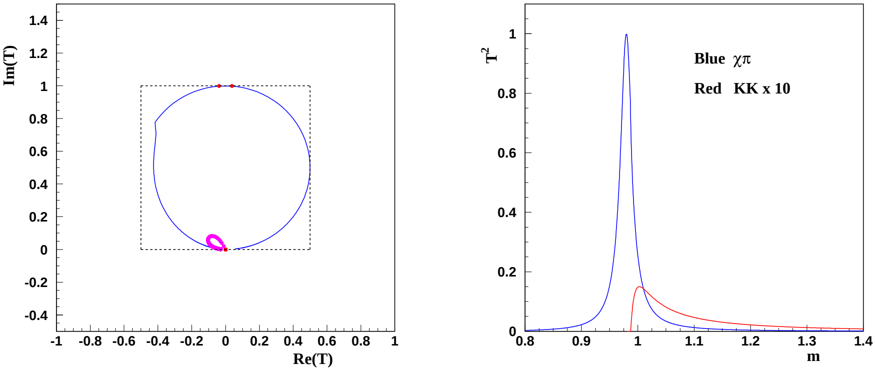
$$T = \frac{m_0\Gamma_0}{m_0^2 - m^2 - im_0\Gamma_0\left(\rho_1\gamma_1^2 + \rho_2\gamma_2^2\right)}\begin{pmatrix} \gamma_1^2 & \gamma_1\gamma_2 \\ \gamma_1\gamma_2 & \gamma_2^2 \end{pmatrix}$$

One-pole, 2-decay K-Matrix



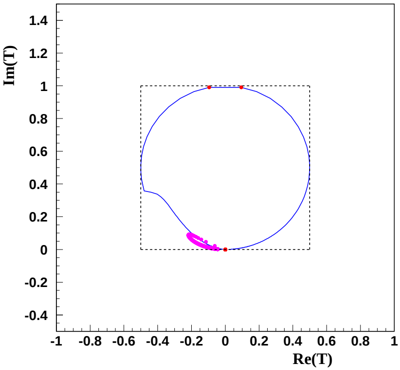
K-matrix mass (0.980) and width (0.080) with $\gamma^2_{\eta\pi} = 0.8$.

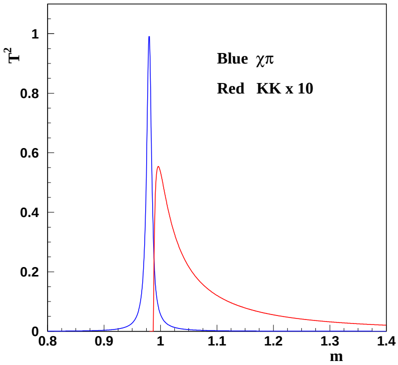
One-pole, 2-decay K-Matrix



K-matrix mass (0.980) and width (0.080) with $\gamma^2_{\eta\pi} = 0.5$.

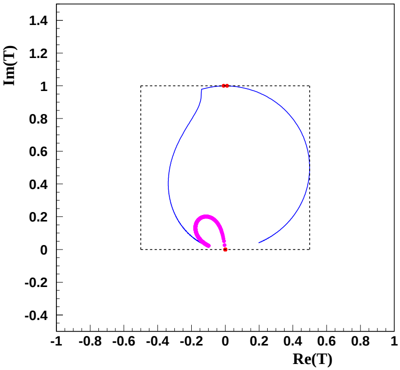
One-pole, 2-decay K-Matrix

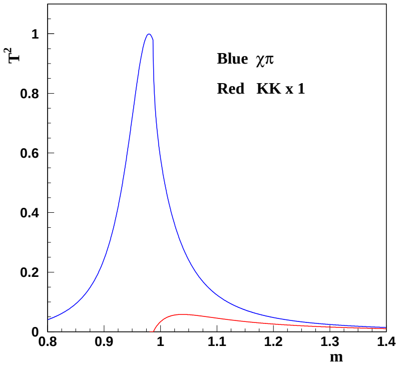




K-matrix mass (0.980) and width (0.080) with $\gamma^2_{\eta\pi} = 0.2$.

One-pole, 2-decay K-Matrix





K-matrix mass (0.980) and width (0.300) with $\gamma^2_{\eta\pi} = 0.5$.

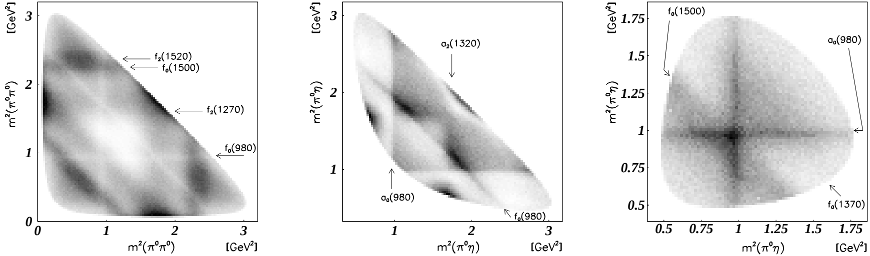
Crystal Barrel Analysis

The scalar mesons, $f_0(1370)$ and $f_0(1500)$ are strongly produced in $\bar{p}p$ annihilation at rest. These can be searched for in three-pseudoscalar final states:

$$
\begin{aligned}
\bar{p}p &\longrightarrow (\pi\pi)\pi^\circ \\
\bar{p}p &\longrightarrow (\pi\pi)\eta \\
\bar{p}p &\longrightarrow (\eta\eta)\pi^\circ \\
\bar{p}p &\longrightarrow (K\bar{K})\pi^\circ \\
\bar{p}p &\longrightarrow (\eta\eta\prime)\pi^\circ
\end{aligned}
$$

The CRYSTAL BARREL Collaboration (C. Amsler *et al.*), *Coupled channel analysis of antiproton proton annihilation into $\pi^\circ\pi^\circ\pi^\circ$, $\eta\eta\pi^\circ$ and $\eta\pi^\circ\pi^\circ$*, Phys. Lett. B**355**, 425, (1995).

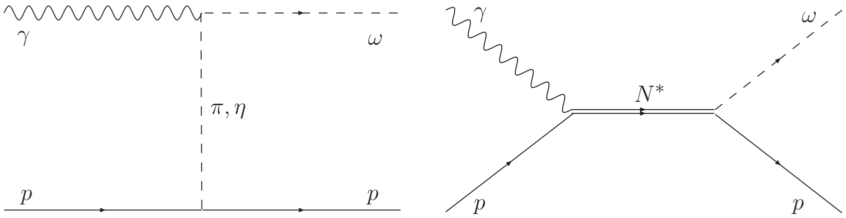
Crystal Barrel Analysis



$$M = (1390 \pm 30) MeV \quad ; \quad \Gamma = (380 \pm 80) MeV$$
$$M = (1500 \pm 10) MeV \quad ; \quad \Gamma = (154 \pm 30) MeV$$

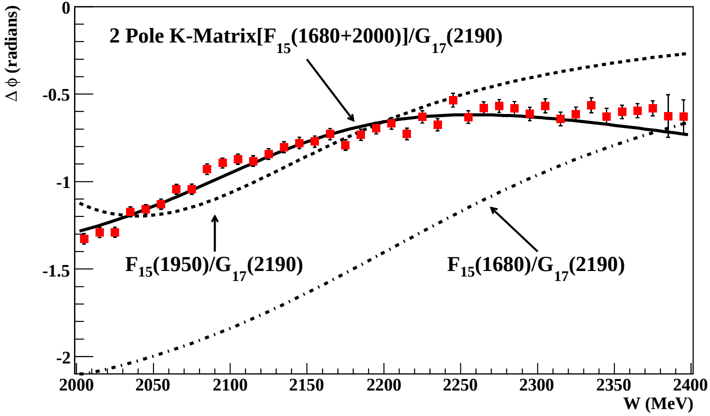
CLAS Photo-production Analysis

The reaction $\gamma p \to p\omega$ from threshold up to $\sqrt{s} \approx 2.8$ GeV. Partial wave analysis carried out which includes both t-channel and s-channel processes.



The CLAS Collaboration (M. Williams *et al.*), *Partial wave analysis of the reaction $\gamma p \to p\omega$ and the search for nucleon resonances*, to be submitted to Phys. Rev. D (2008).

CLAS Photo-production Analysis



Phase difference between the $J^P = (\frac{5}{2})^+$ and the $J^P = (\frac{7}{2})^-$ waves. A reasonable fit to the amplitude and phase differences is obtained by using a two-k-matrix description for the $(\frac{5}{2})^+$ wave.

Summary

- The K-matrix formalism is derived for S-matrix scattering and provides a method to build a unitary T-matrix.
- The formalism accommodates multiple (overlapping) resonances in the same partial wave.
- The formalism allows one to couple data on different final states of the same resonances. This is important when one is trying to measure branching fractions.
- While not mentioned, the extensions to broad daughter particles adds complications to the formalism. In particular, the handling of thresholds and phase-space factors become nebulous.